Fibonacci number

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In mathematics, the **Fibonacci numbers** are the numbers in the following integer sequence:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

(sequence A000045 in OEIS).

By definition, the first two numbers in the Fibonacci sequence are 0 and 1, and each subsequent number is the sum of the previous two.

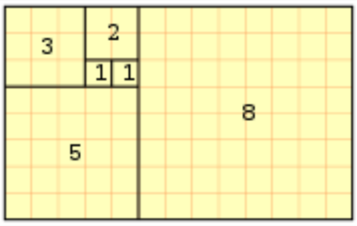
In mathematical terms, the sequence F_n of Fibonacci numbers is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

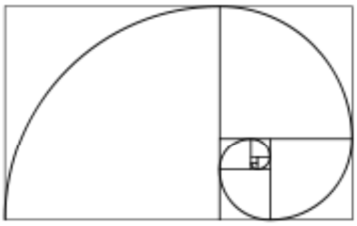
with seed values[1]

$$F_0 = 0,\ F_1 = 1$$

The Fibonacci sequence is named after Leonardo of Pisa, who was known as Fibonacci. Fibonacci's 1202 book *Liber Abaci* introduced the sequence to Western European mathematics,[2] although the sequence had been described earlier in Indian mathematics.[3][4][5] (By modern convention, the sequence begins with $F_0 = 0$. The *Liber Abaci* began the sequence with $F_1 = 1$, omitting the initial 0, and the sequence is still written this way by some.)



A tiling with squares whose sides are successive Fibonacci numbers in length



A Fibonacci spiral created by drawing circular arcs connecting the opposite corners of squares in the Fibonacci tiling; this one uses squares of sizes 1, 1, 2, 3, 5, 8, 13, 21, and 34. See golden spiral.

Fibonacci numbers are closely related to Lucas numbers in that they are a complementary pair of Lucas sequences. They are intimately connected with the golden ratio, for example the closest rational approximations to the ratio are 2/1, 3/2, 5/3, 8/5, ... Applications include computer algorithms such as the Fibonacci search technique and the Fibonacci heap data structure, and graphs called Fibonacci cubes used for interconnecting parallel and distributed systems. They also appear in biological settings,[6] such as branching in trees, arrangement of leaves on a stem, the fruit spouts of a pineapple,[7] the flowering of artichoke, an uncurling fern and the arrangement of a pine cone.[8]

Contents

Origins

The Fibonacci sequence appears in Indian mathematics, in connection with Sanskrit prosody.[4][9] In the Sanskrit oral tradition, there was much emphasis on how long (L) syllables mix with the short (S), and counting the different patterns of L and S within a given fixed length results in the Fibonacci numbers; the number of patterns that are m short syllables long is the Fibonacci number F_{m+1}.[5]

Susantha Goonatilake writes that the development of the Fibonacci sequence "is attributed in part to Pingala (200 BC), later being associated with Virahanka (c. 700 AD), Gopāla (c.1135 AD), and Hemachandra (c.1150)".[3] Parmanand Singh cites Pingala's cryptic formula *misrau cha* ("the two are mixed") and cites scholars who interpret it in context as saying that the cases for m beats (F_{m+1}) is obtained by adding a [S] to F_m cases and [L] to the F_{m-1} cases. He dates Pingala before 450 BCE.[10]

However, the clearest exposition of the series arises in the work of Virahanka (c. 700AD), whose own work is lost, but is available in a quotation by Gopala (c.1135):

> Variations of two earlier meters [is the variation]... For example, for [a meter of length] four, variations of meters of two [and] three being mixed, five happens. [works out examples 8, 13, 21]... In this way, the process should be followed in all *mAtrA-vr.ttas* (prosodic combinations).[11]

The series is also discussed by Gopala (before 1135AD) and by the Jain scholar Hemachandra (c. 1150AD).

In the West, the Fibonacci sequence first appears in the book Liber Abaci (1202) by Leonardo of Pisa, known as Fibonacci.[2] Fibonacci considers the growth of an idealized (biologically unrealistic) rabbit population, assuming that: a newly born pair of rabbits, one male, one female, are put in a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was: how many pairs will there be in one year?

- At the end of the first month, they mate, but there is still only 1 pair.
- At the end of the second month the female produces a new pair, so now there are 2 pairs of rabbits in the field.
- At the end of the third month, the original female produces a second pair, making 3 pairs in all in the field.
- At the end of the fourth month, the original female has produced yet another new pair, the female born two months ago produces her first pair also, making 5 pairs.

At the end of the *n*th month, the number of pairs of rabbits is equal to the number of new pairs (which is the number of pairs in month $n - 2$) plus the number of pairs alive last month ($n - 1$). This is the *n*th Fibonacci number.[12]

The name "Fibonacci sequence" was first used by the 19th-century number theorist Édouard Lucas.[13]

List of Fibonacci numbers

The first 21 Fibonacci numbers F_n for $n = 0, 1, 2, ..., 20$ are:[14]

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	F_{18}	F_{19}	F_{20}
0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	4181	6765

The sequence can also be extended to negative index *n* using the re-arranged recurrence relation

$$F_{n-2} = F_n - F_{n-1},$$

which yields the sequence of "negafibonacci" numbers[15] satisfying

$$F_{-n} = (-1)^{n+1} F_n.$$

Thus the complete sequence is

F_{-8}	F_{-7}	F_{-6}	F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
−21	13	−8	5	−3	2	−1	1	0	1	1	2	3	5	8	13	21

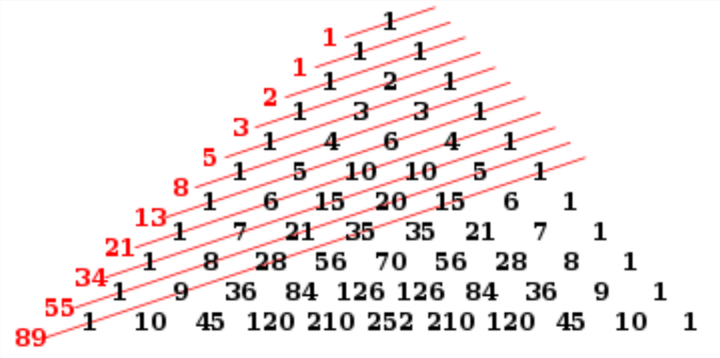
Occurrences in mathematics

The Fibonacci numbers occur in the sums of "shallow" diagonals in Pascal's triangle (*see Binomial coefficient*).[16]

The Fibonacci numbers can be found in different ways in the sequence of binary strings.

- The number of binary strings of length *n* without consecutive 1s is the Fibonacci number F_{n+2}. For example, out of the 16 binary strings of length 4, there are $F_6 = 8$ without consecutive 1s – they

are 0000, 0100, 0010, 0001, 0101, 1000, 1010 and 1001. By symmetry, the number of strings of length *n* without consecutive 0s is also F_{n+2}.
- The number of binary strings of length *n* without an odd number of consecutive 1s is the Fibonacci number F_{n+1}. For example, out of the 16 binary strings of length 4, there are $F_5 = 5$ without an odd number of consecutive 1s – they are 0000, 0011, 0110, 1100, 1111.
- The number of binary strings of length *n* without an even number of consecutive 0s or 1s is $2F_n$. For example, out of the 16 binary strings of length 4, there are $2F_4 = 6$ without an even number of consecutive 0s or 1s – they are 0001, 1000, 1110, 0111, 0101, 1010.



The Fibonacci numbers are the sums of the "shallow" diagonals (shown in red) of Pascal's triangle.

Relation to the golden ratio

Closed-form expression

Like every sequence defined by a linear recurrence with constant coefficients, the Fibonacci numbers have a closed-form solution. It has become known as Binet's formula, even though it was already known by Abraham de Moivre:[17]

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

where

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887\dots$$

is the golden ratio (sequence A001622 in OEIS), and

$$\psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi = -\frac{1}{\varphi}.$$ [18]

To see this,[19] note that φ and ψ are both solutions of the equations

$$x^2 = x + 1, \; x^n = x^{n-1} + x^{n-2},$$

so the powers of φ and ψ satisfy the Fibonacci recursion. In other words

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2}$$

and

$$\psi^n = \psi^{n-1} + \psi^{n-2}.$$

It follows that for any values *a* and *b*, the sequence defined by

$$U_n = a\varphi^n + b\psi^n$$

satisfies the same recurrence

$$U_n = a\varphi^{n-1} + b\psi^{n-1} + a\varphi^{n-2} + b\psi^{n-2} = U_{n-1} + U_{n-2}.$$

If a and b are chosen so that $U_0 = 0$ and $U_1 = 1$ then the resulting sequence U_n must be the Fibonacci sequence. This is the same as requiring a and b satisfy the system of equations:

$$\left\{\begin{array}{l} a+b=0 \\ \varphi a + \psi b = 1 \end{array}\right.$$

which has solution

$$a = \frac{1}{\varphi - \psi} = \frac{1}{\sqrt{5}},\ b = -a$$

producing the required formula.

Computation by rounding

Since

$$\frac{|\psi|^n}{\sqrt{5}} < \frac{1}{2}$$

for all $n \geq 0$, the number F_n is the closest integer to

$$\frac{\varphi^n}{\sqrt{5}}\,.$$

Therefore it can be found by rounding, or in terms of the floor function:

$$F_n = \left\lfloor \frac{\varphi^n}{\sqrt{5}} + \frac{1}{2} \right\rfloor,\ n \geq 0.$$

Similarly, if we already know that the number $F > 1$ is a Fibonacci number, we can determine its index within the sequence by

$$n(F) = \left\lfloor \log_\varphi \left(F \cdot \sqrt{5} + \frac{1}{2} \right) \right\rfloor$$

Limit of consecutive quotients

Johannes Kepler observed that the ratio of consecutive Fibonacci numbers converges. He wrote that "as 5 is to 8 so is 8 to 13, practically, and as 8 is to 13, so is 13 to 21 almost", and concluded that the limit approaches the golden ratio φ.[20]

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$$

This convergence does not depend on the starting values chosen, excluding 0, 0. For example, the initial values 19 and 31 generate the sequence 19, 31, 50, 81, 131, 212, 343, 555 ... etc. The ratio of consecutive terms in this sequence shows the same convergence towards the golden ratio.

In fact this holds for any sequence which satisfies the Fibonacci recurrence other than a sequence of 0's. This can be derived from Binet's formula.

Decomposition of powers of the golden ratio

Since the golden ratio satisfies the equation

$$\varphi^2 = \varphi + 1,$$

this expression can be used to decompose higher powers φ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of φ and 1. The resulting recurrence relationships yield Fibonacci numbers as the linear coefficients:

$$\varphi^n = F(n)\varphi + F(n-1).$$

This expression is also true for $n < 1$ if the Fibonacci sequence $F(n)$ is extended to negative integers using the Fibonacci rule $F(n) = F(n-1) + F(n-2).$

Matrix form

A 2-dimensional system of linear difference equations that describes the Fibonacci sequence is

$$\begin{pmatrix} F_{k+2} \\ F_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$$
$$\vec{F}_{k+1} = A\vec{F}_k$$

The eigenvalues of the matrix A are φ and $(1-\varphi)$, and the elements of the eigenvectors of A, $\binom{\varphi}{1}$ and $\binom{1}{-\varphi}$, are in the ratios φ and $(1-\varphi).$ Using these facts, and the properties of eigenvalues, we can derive a direct formula for the nth element in the Fibonacci series:

$$F_n = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

The matrix has a determinant of −1, and thus it is a 2×2 unimodular matrix. This property can be understood in terms of the continued fraction representation for the golden ratio:

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$$

The Fibonacci numbers occur as the ratio of successive convergents of the continued fraction for φ, and the matrix formed from successive convergents of any continued fraction has a determinant of +1 or −1.

The matrix representation gives the following closed expression for the Fibonacci numbers:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Taking the determinant of both sides of this equation yields Cassini's identity

$$(-1)^n = F_{n+1}F_{n-1} - F_n^2 .$$

Additionally, since $A^nA^m = A^{m+n}$ for any square matrix *A*, the following identities can be derived:

$$\begin{aligned} F_mF_n + F_{m-1}F_{n-1} &= F_{m+n-1} \\ F_{n+1}F_m + F_nF_{m-1} &= F_{m+n} \end{aligned}$$

In particular, with $m = n$,

$$\begin{aligned} F_{2n-1} &= F_n^2 + F_{n-1}^2 \\ F_{2n} &= (F_{n-1}+F_{n+1})F_n \\ &= (2F_{n-1}+F_n)F_n \end{aligned}$$

Recognizing Fibonacci numbers

The question may arise whether a positive integer *z* is a Fibonacci number. Since $F(n)$ is the closest integer to $\varphi^n/\sqrt{5}$, the most straightforward, brute-force test is the identity

$$F\left(\bigg\lfloor \log_\varphi \left(z\cdot\sqrt{5}+\frac{1}{2}\right)\bigg\rfloor\right) = z,$$

which is true if and only if *z* is a Fibonacci number. In this formula, $F(n)$ can be computed rapidly using any of the previously discussed closed-form expressions.

One implication of the above expression is this: if it is known that a number *z* is a Fibonacci number, we may determine an *n* such that $F(n) = z$ by the following:

$$\bigg\lfloor \log_\varphi \left(z\cdot\sqrt{5}+\frac{1}{2}\right)\bigg\rfloor = n$$

Alternatively, a positive integer *z* is a Fibonacci number if and only if one of $5z^2+4$ or $5z^2-4$ is a perfect square.[21]

A slightly more sophisticated test uses the fact that the convergents of the continued fraction representation of φ are ratios of successive Fibonacci numbers. That is, the inequality

$$\left|\varphi - \frac{p}{q}\right| < \frac{1}{q^2}$$

(with coprime positive integers *p*, *q*) is true if and only if *p* and *q* are successive Fibonacci numbers. From this one derives the criterion that *z* is a Fibonacci number if and only if the closed interval

$$\left[\varphi z - \frac{1}{z}, \varphi z + \frac{1}{z}\right]$$

contains a positive integer.[22] For $z \geq 2$, it is easy to show that this interval contains at most one integer, and in the event that *z* is a Fibonacci number, the contained integer is equal to the next successive Fibonacci number after *z*. Somewhat remarkably, this result still holds for the case $z = 1$, but it must be stated carefully since 1 appears twice in the Fibonacci sequence, and thus has two distinct successors.

Identities

Most identities involving Fibonacci numbers draw from combinatorial arguments. $F(n)$ can be interpreted as the number of sequences of 1s and 2s that sum to $n - 1$, with the convention that $F(0) = 0$, meaning no sum will add up to −1, and that $F(1) = 1$, meaning the empty sum will "add up" to 0. Here the order of the summands matters. For example, 1 + 2 and 2 + 1 are considered two different sums and are counted twice.

First identity

$$F_n = F_{n-1} + F_{n-2}$$

For $n > 1$.

The nth Fibonacci number is the sum of the previous two Fibonacci numbers.

Proof

We must establish that the sequence of numbers defined by the combinatorial interpretation above satisfy the same recurrence relation as the Fibonacci numbers (and so are indeed identical to the Fibonacci numbers).

The set of $F(n + 1)$ ways of making ordered sums of 1s and 2s that sum to n may be divided into two non-overlapping sets. The first set contains those sums whose first summand is 1; the remainder sums to $n - 1$, so there are $F(n)$ sums in the first set. The second set contains those sums whose first summand is 2; the remainder sums to $n - 2$, so there are $F(n - 1)$ sums in the second set. The first summand can only be 1 or 2, so these two sets exhaust the original set. Thus $F(n + 1) = F(n) + F(n - 1)$.

Second identity

The sum of the first n Fibonacci numbers is equal to the n+2nd Fibonacci number minus 1.[23] In symbols:

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1$$

The sum of the first n *Fibonacci numbers is the (*n *+ 2)nd Fibonacci number minus 1.*

Proof

We count the number of ways summing 1s and 2s to $n + 1$ such that at least one of the summands is 2.

As before, there are $F(n + 2)$ ways summing 1s and 2s to $n + 1$ when $n \geq 0$. Since there is only one sum of $n + 1$ that does not use any 2, namely $1 + ... + 1$ ($n + 1$ terms), we subtract 1 from $F(n + 2)$.

Equivalently, we can consider the first occurrence of 2 as a summand. If, in a sum, the first summand is 2, then there are $F(n)$ ways to the complete the counting for $n - 1$. If the second summand is 2 but the first is 1, then there are $F(n - 1)$ ways to complete the counting for $n - 2$. Proceed in this fashion. Eventually we consider the $(n + 1)$th summand. If it is 2 but all of the previous n summands are 1s, then there are $F(0)$ ways to complete the counting for 0. If a sum contains 2 as a summand, the first occurrence of such summand must take place in between the first and $(n + 1)$th position. Thus $F(n) + F(n - 1) + ... + F(0)$ gives the desired counting.

By induction:

For $n = 0$, $\sum_{i=0}^{0} F_i = F_2 - 1 = 1 - 1 = 0$, so the equation is true for $n = 0$.

For $n = x$, assume $\sum_{i=0}^{x} F_i = F_{x+2} - 1$.

Add the next Fibonacci number F_{x+1} to both sides: $F_{x+1} + \sum_{i=0}^{x} F_i = F_{x+1} + F_{x+2} - 1$.

By the Fibonacci recurrence relation, $F_{x+1} + F_{x+2} = F_{x+3}$, so $\sum_{i=0}^{x+1} F_i = F_{x+3} - 1$, which is the $n = x + 1$ case, proving that where the equation is true for $n = x$, so is it for $n = x + 1$.

Third identity

This identity has slightly different forms for F_j, depending on whether j is odd or even.

The sum of the first n − 1 Fibonacci numbers, F_j, such that j is odd, is the (2n)th Fibonacci number.

$$\sum_{i=0}^{n-1} F_{2i+1} = F_{2n}$$

The sum of the first n Fibonacci numbers, F_j, such that j is even, is the (2n + 1)th Fibonacci number minus 1.

$$\sum_{i=0}^{n} F_{2i} = F_{2n+1} - 1$$

[24]

Proofs

1: j is odd

By induction for F_{2n}:

$$\begin{aligned} F_1 + F_3 + F_5 + \cdots + F_{2n-3} + F_{2n-1} &= F_{2n} \\ F_1 + F_3 + F_5 + \cdots + F_{2n-3} + F_{2n-1} + F_{2n+1} &= F_{2n} + F_{2n+1} \\ F_1 + F_3 + F_5 + \cdots + F_{2n-3} + F_{2n-1} + F_{2n+1} &= F_{2n+2} \end{aligned}$$

A basis case for this could be $F_1 = F_2$.

2: j is even

By induction for F_{2n+1}:

$$\begin{aligned} F_0 + F_2 + F_4 + \cdots + F_{2n-2} + F_{2n} &= F_{2n+1} - 1 \\ F_0 + F_2 + F_4 + \cdots + F_{2n-2} + F_{2n} + F_{2n+2} &= F_{2n+1} + F_{2n+2} - 1 \\ F_0 + F_2 + F_4 + \cdots + F_{2n-2} + F_{2n} + F_{2n+2} &= F_{2n+3} - 1 \end{aligned}$$

A basis case for this could be $F_0 = F_1 - 1$.

Alternative proof

By using identity 1 we can construct a telescoping sum:

$$\sum_{i=0}^{n-1} F_{2i+1} = \sum_{i=0}^{n-1} [F_{2(i+1)} - F_{2i}] = F_{2n} - F_0 = F_{2n}$$

If the summands are the Fibonacci numbers with even index, the proof is very similar. Summing both cases yields identity 2.

Fourth identity

$$\sum_{i=0}^{n} iF_i = nF_{n+2} - F_{n+3} + 2$$

Proof

This identity can be established in two stages. First, we count the number of ways summing 1s and 2s to −1, 0, ..., or $n + 1$ such that at least one of the summands is 2.

By our second identity, there are $F(n + 2) - 1$ ways summing to $n + 1$; $F(n + 1) - 1$ ways summing to n; ...; and, eventually, $F(2) - 1$ way summing to 1. As $F(1) - 1 = F(0) = 0$, we can add up all $n + 1$ sums and apply the second identity again to obtain

$$\begin{aligned}
&[F(n + 2) - 1] + [F(n + 1) - 1] + ... + [F(2) - 1] \\
&= [F(n + 2) - 1] + [F(n + 1) - 1] + ... + [F(2) - 1] + [F(1) - 1] + F(0) \\
&= F(n + 2) + [F(n + 1) + ... + F(1) + F(0)] - (n + 2) \\
&= F(n + 2) + [F(n + 3) - 1] - (n + 2) \\
&= F(n + 2) + F(n + 3) - (n + 3).
\end{aligned}$$

On the other hand, we observe from the second identity that there are

- $F(0) + F(1) + ... + F(n - 1) + F(n)$ ways summing to $n + 1$;
- $F(0) + F(1) + ... + F(n - 1)$ ways summing to n;

......

- $F(0)$ way summing to −1.

Adding up all $n + 1$ sums, we see that there are

- $(n + 1)\, F(0) + n\, F(1) + ... + F(n)$ ways summing to −1, 0, ..., or $n + 1$.

Since the two methods of counting refer to the same number, we have

$$(n + 1)\, F(0) + n\, F(1) + ... + F(n) = F(n + 2) + F(n + 3) - (n + 3)$$

Finally, we complete the proof by subtracting the above identity from $n + 1$ times the second identity.

Fifth identity

$$\sum_{i=0}^{n} {F_i}^2 = F_n F_{n+1}$$

The sum of the squares of the first n Fibonacci numbers is the product of the nth and (n + 1)th Fibonacci numbers.

Proof

Although this identity can be established by either induction or direct, albeit messy, algebraic manipulation, perhaps the most elegant and most insightful method is by a simple geometric argument.

Consider the Fibonacci Rectangles constructed in previous sections. Using a common trick, we will compute the area of this rectangle in two different ways. But since this must yield the same answer in both cases, we know these resulting expressions must be equal, which will yield the desired identity.

On the one hand, the n-th rectangle is composed of n squares, whose side lengths are $F(1)$, $F(2)$, ..., $F(n)$. Its area is therefore the sum of each of these squares, which is given by

$$\sum_{i=0}^{n} {F_i}^2.$$

On the other hand, we know that the n-th rectangle has side lengths $F(n)$ and $F(n + 1)$. Thus, its area is simply given by

$$F_n F_{n+1}.$$

Setting these expressions equal to each other completes the proof.

Identity for doubling *n*

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2 = F_n(F_{n+1} + F_{n-1}) = F_n L_n$$ [25]

Where L_n is the n'th Lucas Number.

Another identity

Another identity useful for calculating F_n for large values of n is[25]

$$F_{kn+c} = \sum_{i=0}^{k} \binom{k}{i} F_{c-i} F_n^i F_{n+1}^{k-i},$$

from which other identities for specific values of k, n, and c can be derived below, including

$$F_{2n+k} = F_k F_{n+1}^2 + 2F_{k-1} F_{n+1} F_n + F_{k-2} F_n^2$$

for all integers n and k. Doubling identities of this type can be used to calculate F_n using O(log n) long multiplication operations of size n bits. The number of bits of precision needed to perform each multiplication doubles at each step, so the performance is limited by the final multiplication; if the fast Schönhage–Strassen multiplication algorithm is used, this is O(n log n log log n) bit operations. Notice that, with the definition of Fibonacci numbers with negative n given in the introduction, this formula reduces to the *double n* formula when $k = 0$.

Other identities

Other identities include relationships to the Lucas numbers, which have the same recursive properties but start with $L_0 = 2$ and $L_1 = 1$. These properties include $F_{2n} = F_n L_n$.

There are also scaling identities, which take you from F_n and F_{n+1} to a variety of things of the form F_{an+b}; for instance

$$F_{3n} = 2F_n^3 + 3F_nF_{n+1}F_{n-1} = 5F_n^3 + 3(-1)^nF_n$$ by Cassini's identity.

$$F_{3n+1} = F_{n+1}^3 + 3F_{n+1}F_n^2 - F_n^3$$

$$F_{3n+2} = F_{n+1}^3 + 3F_{n+1}^2F_n + F_n^3$$

$$F_{4n} = 4F_nF_{n+1}(F_{n+1}^2 + 2F_n^2) - 3F_n^2(F_n^2 + 2F_{n+1}^2)$$

These can be found experimentally using lattice reduction, and are useful in setting up the special number field sieve to factorize a Fibonacci number. Such relations exist in a very general sense for numbers defined by recurrence relations. See the section on multiplication formulae under Perrin numbers for details.

Power series

The generating function of the Fibonacci sequence is the power series

$$s(x) = \sum_{k=0}^{\infty} F_k x^k.$$

This series has a simple and interesting closed-form solution for $|x| < \frac{1}{\varphi}$:[26]

$$s(x) = \frac{x}{1 - x - x^2}.$$

This solution can be proven by using the Fibonacci recurrence to expand each coefficient in the infinite sum defining $s(x)$:

$$\begin{aligned}s(x) &= \sum_{k=0}^{\infty} F_k x^k \\ &= F_0 + F_1 x + \sum_{k=2}^{\infty} \left(F_{k-1} + F_{k-2}\right) x^k \\ &= x + \sum_{k=2}^{\infty} F_{k-1} x^k + \sum_{k=2}^{\infty} F_{k-2} x^k \\ &= x + x\sum_{k=0}^{\infty} F_k x^k + x^2 \sum_{k=0}^{\infty} F_k x^k \\ &= x + x s(x) + x^2 s(x).\end{aligned}$$

Solving the equation $s(x) = x + xs(x) + x^2 s(x)$ for $s(x)$ results in the closed form solution.

In particular, math puzzle-books note the curious value $\frac{s(\frac{1}{10})}{10} = \frac{1}{89}$,[27] or more generally

$$\sum_{n=1}^{\infty} \frac{F_n}{10^{(k+1)(n+1)}} = \frac{1}{10^{2k+2} - 10^{k+1} - 1}$$

for all integers $k \geq 0$.

More generally,

$$\sum_{n=0}^{\infty} \frac{F_n}{k^n} = \frac{k}{k^2 - k - 1}.$$

Reciprocal sums

Infinite sums over reciprocal Fibonacci numbers can sometimes be evaluated in terms of theta functions. For example, we can write the sum of every odd-indexed reciprocal Fibonacci number as

$$\sum_{k=0}^{\infty} \frac{1}{F_{2k+1}} = \frac{\sqrt{5}}{4} \vartheta_2^2 \left(0, \frac{3-\sqrt{5}}{2}\right),$$

and the sum of squared reciprocal Fibonacci numbers as

$$\sum_{k=1}^{\infty} \frac{1}{F_k^2} = \frac{5}{24} \left(\vartheta_2^4 \left(0, \frac{3-\sqrt{5}}{2}\right) - \vartheta_4^4 \left(0, \frac{3-\sqrt{5}}{2}\right) + 1\right).$$

If we add 1 to each Fibonacci number in the first sum, there is also the closed form

$$\sum_{k=0}^{\infty} \frac{1}{1 + F_{2k+1}} = \frac{\sqrt{5}}{2},$$

and there is a nice *nested* sum of squared Fibonacci numbers giving the reciprocal of the golden ratio,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sum_{j=1}^{k} {F_j}^2} = \frac{\sqrt{5}-1}{2}.$$

Results such as these make it plausible that a closed formula for the plain sum of reciprocal Fibonacci numbers could be found, but none is yet known. Despite that, the reciprocal Fibonacci constant

$$\psi = \sum_{k=1}^{\infty} \frac{1}{F_k} = 3.359885666243\dots$$

has been proved irrational by Richard André-Jeannin.

Millin series gives a remarkable identity:[28]

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}$$

which follows from the closed form for its partial sums as N tends to infinity:

$$\sum_{n=0}^{N} \frac{1}{F_{2^n}} = 3 - \frac{F_{2^N-1}}{F_{2^N}}.$$

Primes and divisibility

Divisibility properties

Every 3rd number of the sequence is even and more generally, every kth number of the sequence is a multiple of F_k. Thus the Fibonacci sequence is an example of a divisibility sequence. In fact, the Fibonacci sequence satisfies the stronger divisibility property

$$\gcd(F_m, F_n) = F_{\gcd(m,n)}\ .$$

Fibonacci primes

Main article: Fibonacci prime

A *Fibonacci prime* is a Fibonacci number that is prime. The first few are:

2, 3, 5, 13, 89, 233, 1597, 28657, 514229, … (sequence A005478 in OEIS).

Fibonacci primes with thousands of digits have been found, but it is not known whether there are infinitely many.[29]

F_{kn} is divisible by F_n, so, apart from $F_4 = 3$, any Fibonacci prime must have a prime index. As there are arbitrarily long runs of composite numbers, there are therefore also arbitrarily long runs of composite Fibonacci numbers.

With the exceptions of 1, 8 and 144 ($F_1 = F_2$, F_6 and F_{12}) every Fibonacci number has a prime factor that is not a factor of any smaller Fibonacci number (Carmichael's theorem).[30]

144 is the only nontrivial square Fibonacci number.[31] Attila Pethő proved[32] in 2001 that there are only finitely many perfect power Fibonacci numbers. In 2006, Y. Bugeaud, M. Mignotte, and S. Siksek proved

that only 8 and 144 are non-trivial perfect powers.[33]

No Fibonacci number greater than $F_6 = 8$ is one greater or one less than a prime number.[34]

Any three consecutive Fibonacci numbers, taken two at a time, are relatively prime: that is,

$$\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = 1.$$

More generally,

$$\gcd(F_n, F_m) = F_{\gcd(n,m)}.$$[35][36]

Prime divisors of Fibonacci numbers

The divisibility of Fibonacci numbers by a prime p is related to the Legendre symbol $\left(\tfrac{p}{5}\right)$ which is evaluated as follows:

$$\left(\frac{p}{5}\right) = \begin{cases} 0 & \text{if } p = 5 \\ 1 & \text{if } p \equiv \pm 1 \pmod 5 \\ -1 & \text{if } p \equiv \pm 2 \pmod 5. \end{cases}$$

If p is a prime number then $F_p \equiv \left(\frac{p}{5}\right) \pmod p \quad \text{and} \quad F_{p-\left(\frac{p}{5}\right)} \equiv 0 \pmod p.$[37][38]

For example,

$$\begin{aligned} \left(\tfrac{2}{5}\right) &= -1, & F_3 &= 2, & F_2 &= 1, \\ \left(\tfrac{3}{5}\right) &= -1, & F_4 &= 3, & F_3 &= 2, \\ \left(\tfrac{5}{5}\right) &= 0, & F_5 &= 5, & & \\ \left(\tfrac{7}{5}\right) &= -1, & F_8 &= 21, & F_7 &= 13, \\ \left(\tfrac{11}{5}\right) &= +1, & F_{10} &= 55, & F_{11} &= 89. \end{aligned}$$

It is not known whether there exists a prime p such that $F_{p-\left(\frac{p}{5}\right)} \equiv 0 \pmod{p^2}$. Such primes (if there are any) would be called Wall–Sun–Sun primes.

Also, if $p \neq 5$ is an odd prime number then:[39]

$$5F^2_{(p\pm1)/2} \equiv \begin{cases} \frac{5\left(\frac{p}{5}\right)\pm 5}{2} \pmod p & \text{if } p \equiv 1 \pmod 4 \\ \\ \frac{5\left(\frac{p}{5}\right)\mp 3}{2} \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Examples of all the cases:

$$p = 7 \equiv 3 \pmod 4, \ \left(\tfrac{7}{5}\right) = -1, \frac{5\left(\frac{7}{5}\right)+3}{2} = -1 \text{ and } \frac{5\left(\frac{7}{5}\right)-3}{2} = -4.$$

$$F_3 = 2 \text{ and } F_4 = 3.$$

$$5F_3^2 = 20 \equiv -1 \pmod 7 \quad \text{and} \quad 5F_4^2 = 45 \equiv -4 \pmod 7$$

$$p = 11 \equiv 3 \pmod 4, \ (\tfrac{11}{5}) = +1, \frac{5(\frac{11}{5})+3}{2} = 4 \text{ and } \frac{5(\frac{11}{5})-3}{2} = 1.$$

$$F_5 = 5 \text{ and } F_6 = 8.$$

$$5F_5^2 = 125 \equiv 4 \pmod{11} \quad \text{and} \quad 5F_6^2 = 320 \equiv 1 \pmod{11}$$

$$p = 13 \equiv 1 \pmod 4, \ (\tfrac{13}{5}) = -1, \frac{5(\frac{13}{5})-5}{2} = -5 \text{ and } \frac{5(\frac{13}{5})+5}{2} = 0.$$

$$F_6 = 8 \text{ and } F_7 = 13.$$

$$5F_6^2 = 320 \equiv -5 \pmod{13} \quad \text{and} \quad 5F_7^2 = 845 \equiv 0 \pmod{13}$$

$$p = 29 \equiv 1 \pmod 4, \ (\tfrac{29}{5}) = +1, \frac{5(\frac{29}{5})-5}{2} = 0 \text{ and } \frac{5(\frac{29}{5})+5}{2} = 5.$$

$$F_{14} = 377 \text{ and } F_{15} = 610.$$

$$5F_{14}^2 = 710645 \equiv 0 \pmod{29} \quad \text{and} \quad 5F_{15}^2 = 1860500 \equiv 5 \pmod{29}$$

For odd n, all odd prime divisors of F_n are ≡ 1 (mod 4), implying that all odd divisors of F_n (as the products of odd prime divisors) are ≡ 1 (mod 4).[40]

> For example, $F_1 = 1$, $F_3 = 2$, $F_5 = 5$, $F_7 = 13$, $F_9 = 34 = 2×17$, $F_{11} = 89$, $F_{13} = 233$, $F_{15} = 610 = 2×5×61$

All known factors of Fibonacci numbers F(i) for all i < 50000 are collected at the relevant repositories[41][42].

Periodicity modulo *n*

Main article: Pisano period

It may be seen that if the members of the Fibonacci sequence are taken mod n, the resulting sequence must be periodic with period at most n^2-1. The lengths of the periods for various n form the so-called Pisano periods (sequence A001175 in OEIS). Determining the Pisano periods in general is an open problem, [*citation needed*] although for any particular n it can be solved as an instance of cycle detection.

Right triangles

Starting with 5, every second Fibonacci number is the length of the hypotenuse of a right triangle with integer sides, or in other words, the largest number in a Pythagorean triple. The length of the longer leg of this triangle is equal to the sum of the three sides of the preceding triangle in this series of triangles, and the shorter leg is equal to the difference between the preceding bypassed Fibonacci number and the shorter leg of the preceding triangle.

The first triangle in this series has sides of length 5, 4, and 3. Skipping 8, the next triangle has sides of length 13, 12 (5 + 4 + 3), and 5 (8 − 3). Skipping 21, the next triangle has sides of length 34, 30 (13 + 12 + 5), and 16 (21 − 5). This series continues indefinitely. The triangle sides a, b, c can be calculated directly:

$$\begin{aligned} a_n &= F_{2n-1} \\ b_n &= 2F_nF_{n-1} \\ c_n &= F_n^2 - F_{n-1}^2. \end{aligned}$$

These formulas satisfy $a_n^2 = b_n^2 + c_n^2$ for all n, but they only represent triangle sides when $n > 2$.

Any four consecutive Fibonacci numbers F_n, F_{n+1}, F_{n+2} and F_{n+3} can also be used to generate a Pythagorean triple in a different way[43]:

$$a = F_n F_{n+3}\,;\; b = 2F_{n+1}F_{n+2}\,;\; c = F_{n+1}^2 + F_{n+2}^2\,;\; a^2 + b^2 = c^2\,.$$

Example 1: let the Fibonacci numbers be 1, 2, 3 and 5. Then:

$$\begin{aligned}&a = 1 \times 5 = 5\\&b = 2 \times 2 \times 3 = 12\\&c = 2^2 + 3^2 = 13\\&5^2 + 12^2 = 13^2\,.\end{aligned}$$

Magnitude

Since F_n is asymptotic to $\varphi^n/\sqrt{5}$, the number of digits in F_n is asymptotic to $n\ \log_{10}\varphi \approx 0.2090\,n$. As a consequence, for every integer $d > 1$ there are either 4 or 5 Fibonacci numbers with d decimal digits.

More generally, in the base b representation, the number of digits in F_n is asymptotic to $n\ \log_b \varphi$.

Applications

The Fibonacci numbers are important in the computational run-time analysis of Euclid's algorithm to determine the greatest common divisor of two integers: the worst case input for this algorithm is a pair of consecutive Fibonacci numbers.[44]

Yuri Matiyasevich was able to show that the Fibonacci numbers can be defined by a Diophantine equation, which led to his original solution of Hilbert's tenth problem.

The Fibonacci numbers are also an example of a complete sequence. This means that every positive integer can be written as a sum of Fibonacci numbers, where any one number is used once at most. Specifically, every positive integer can be written in a unique way as the sum of *one or more* distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. This is known as Zeckendorf's theorem, and a sum of Fibonacci numbers that satisfies these conditions is called a Zeckendorf representation. The Zeckendorf representation of a number can be used to derive its Fibonacci coding.

Fibonacci numbers are used by some pseudorandom number generators.

Fibonacci numbers are used in a polyphase version of the merge sort algorithm in which an unsorted list is divided into two lists whose lengths correspond to sequential Fibonacci numbers – by dividing the list so that the two parts have lengths in the approximate proportion φ. A tape-drive implementation of the polyphase merge sort was described in *The Art of Computer Programming*.

Fibonacci numbers arise in the analysis of the Fibonacci heap data structure.

The Fibonacci cube is an undirected graph with a Fibonacci number of nodes that has been proposed as a network topology for parallel computing.

A one-dimensional optimization method, called the Fibonacci search technique, uses Fibonacci numbers.[45]

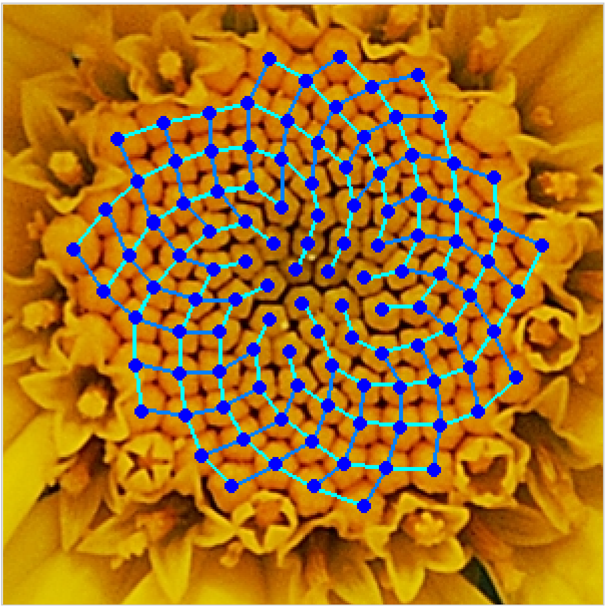
The Fibonacci number series is used for optional lossy compression in the IFF 8SVX audio file format used on Amiga computers. The number series compands the original audio wave similar to logarithmic methods such as μ-law.[46][47]

In music, Fibonacci numbers are sometimes used to determine tunings, and, as in visual art, to determine the length or size of content or formal elements. It is commonly thought that the third movement of Béla Bartók's *Music for Strings, Percussion, and Celesta* was structured using Fibonacci numbers.

Since the conversion factor 1.609344 for miles to kilometers is close to the golden ratio (denoted φ), the decomposition of distance in miles into a sum of Fibonacci numbers becomes nearly the kilometer sum when the Fibonacci numbers are replaced by their successors. This method amounts to a radix 2 number register in golden ratio base φ being shifted. To convert from kilometers to miles, shift the register down the Fibonacci sequence instead.[48]

In nature

Fibonacci sequences appear in biological settings,[6] in two consecutive Fibonacci numbers, such as branching in trees, arrangement of leaves on a stem, the fruitlets of a pineapple,[7] the flowering of artichoke, an uncurling fern and the arrangement of a pine cone.[8] In addition, numerous poorly substantiated claims of Fibonacci numbers or golden sections in nature are found in popular sources, e.g., relating to the breeding of rabbits, the seeds on a sunflower, the spirals of shells, and the curve of waves.[49] The Fibonacci numbers are also found in the family tree of honeybees.[50]



Yellow Chamomile head showing the arrangement in 21 (blue) and 13 (aqua) spirals. Such arrangements involving consecutive Fibonacci numbers appear in a wide variety of plants.

Przemysław Prusinkiewicz advanced the idea that real instances can in part be understood as the expression of certain algebraic constraints on free groups, specifically as certain Lindenmayer grammars.[51]

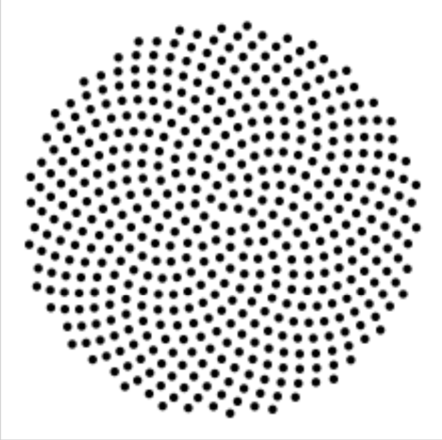


Illustration of Vogel's model for n=1 ... 500

A model for the pattern of florets in the head of a sunflower was proposed by H. Vogel in 1979.[52] This has the form

$$\theta = \frac{2\pi}{\phi^2} n, \ r = c\sqrt{n}$$

where n is the index number of the floret and c is a constant scaling factor; the florets thus lie on Fermat's spiral. The divergence angle, approximately 137.51°, is the golden angle, dividing the circle in the golden ratio. Because this ratio is irrational, no floret has a neighbor at exactly the same angle from the center, so the florets pack efficiently. Because the rational approximations to the golden ratio are of the form $F(j):F(j+1)$, the nearest neighbors of floret number n are those at $n \pm F(j)$ for some index j which depends on r, the distance from the center. It is often said that sunflowers and similar arrangements have 55 spirals in one direction and 89 in the other (or some other pair of adjacent Fibonacci numbers), but this is true only of one range of radii, typically the outermost and thus most conspicuous.[53]

The bee ancestry code

Fibonacci numbers also appear in the description of the reproduction of a population of idealized honeybees, according to the following rules:

- If an egg is laid by an unmated female, it hatches a male or drone bee.
- If, however, an egg was fertilized by a male, it hatches a female.

Thus, a male bee will always have one parent, and a female bee will have two.

If one traces the ancestry of any male bee (1 bee), he has 1 parent (1 bee), 2 grandparents, 3 great-grandparents, 5 great-great-grandparents, and so on. This sequence of numbers of parents is the Fibonacci sequence. The number of ancestors at each level, F_n, is the number of female ancestors, which is F_{n-1}, plus the number of male ancestors, which is F_{n-2}.[54] (This is under the unrealistic assumption that the ancestors at each level are otherwise unrelated.)

Popular culture

Main article: Fibonacci numbers in popular culture

Generalizations

Main article: Generalizations of Fibonacci numbers

The Fibonacci sequence has been generalized in many ways. These include:

- Generalizing the index to negative integers to produce the Negafibonacci numbers.
- Generalizing the index to real numbers using a modification of Binet's formula.[25]
- Starting with other integers. Lucas numbers have $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$. Primefree sequences use the Fibonacci recursion with other starting points in order to generate sequences in which all numbers are composite.
- Letting a number be a linear function (other than the sum) of the 2 preceding numbers. The Pell numbers have $P_n = 2P_{n-1} + P_{n-2}$.
- Not adding the immediately preceding numbers. The Padovan sequence and Perrin numbers have $P(n) = P(n-2) + P(n-3)$.
- Generating the next number by adding 3 numbers (tribonacci numbers), 4 numbers (tetranacci numbers), or more. The resulting sequences are known as *n-Step Fibonacci numbers*.[55]
- Adding other objects than integers, for example functions or strings—one essential example is Fibonacci polynomials.

See also

- Collatz conjecture
- Fibonacci word
- Helicoid
- Lucas numbers
- The Fibonacci Association
- Recursion (computer science)#Fibonacci

Notes

1. ^ Lucas p. 3
2. ^ ***a*** ***b*** Sigler, Laurence E. (trans.) (2002). *Fibonacci's Liber Abaci*. Springer-Verlag. ISBN 0-387-95419-8.

Chapter II.12, pp. 404–405.

3. ^ ***a*** ***b*** Susantha Goonatilake (1998). *Toward a Global Science* (http://books.google.com/?id=SI5ip95BbgEC&pg=PA126&dq=Virahanka+Fibonacci) . Indiana University Press. p. 126. ISBN 9780253333889. http://books.google.com/?id=SI5ip95BbgEC&pg=PA126&dq=Virahanka+Fibonacci.
4. ^ ***a*** ***b*** Singh, Parmanand (1985). "The So-called Fibonacci numbers in ancient and medieval India". *Historia Mathematica* **12** (3): 229–244. doi:10.1016/0315-0860(85)90021-7 (http://dx.doi.org/10.1016%2F0315-0860%2885%2990021-7) .
5. ^ ***a*** ***b*** Donald Knuth (2006). *The Art of Computer Programming: Generating All Trees—History of Combinatorial Generation; Volume 4* (http://books.google.com/?id=56LNfE2QGtYC&pg=PA50&dq=rhythms) . Addison–Wesley. p. 50. ISBN 9780321335708. http://books.google.com/?id=56LNfE2QGtYC&pg=PA50&dq=rhythms. quote: it was natural to consider the set of all sequences of [L] and [S] that have exactly m beats. ... there are exactly Fm+1 of them. For example the 21 sequences when $m = 7$ are: [gives list]. In this way Indian prosodists were led to discover the Fibonacci sequence, as we have observed in Section 1.2.8 (from v.1)
6. ^ ***a*** ***b*** S. Douady and Y. Couder (1996). "Phyllotaxis as a Dynamical Self Organizing Process" (http://www.math.ntnu.no/~jarlet/Douady96.pdf) (PDF). *Journal of Theoretical Biology* **178** (178): 255–274. doi:10.1006/jtbi.1996.0026 (http://dx.doi.org/10.1006%2Fjtbi.1996.0026) . http://www.math.ntnu.no/~jarlet/Douady96.pdf.
7. ^ ***a*** ***b*** Jones, Judy; William Wilson (2006). "Science". *An Incomplete Education*. Ballantine Books. p. 544. ISBN 978-0-7394-7582-9.
8. ^ ***a*** ***b*** A. Brousseau (1969). "Fibonacci Statistics in Conifers". *Fibonacci Quarterly* (7): 525–532.
9. ^ Donald Knuth (1968). *The Art Of Computer Programming, Volume 1* (http://books.google.com/?id=MooMkK6ERuYC&pg=PA100&dq=knuth+gopala+fibonacci#v=onepage&) . Addison Wesley. ISBN 8177587544. http://books.google.com/?id=MooMkK6ERuYC&pg=PA100&dq=knuth+gopala+fibonacci#v=onepage&.quote: "Before Fibonacci wrote his work, the sequence Fn had already been discussed by Indian scholars, who had long been interested in rhythmic patterns... both Gopala (before 1135AD) and Hemachandra (c.1150) mentioned the numbers 1,2,3,5,8,13,21 explicitly. [See P. Singh Historia Math 12 (1985) 229–244]" p. 100 (3d ed)...
10. ^ [Agrawala. V. S. 1969]. pAninikAlIna bhAratavarSha (Hn.). Varanasi-I: TheChowkhamba Vidyabhawan.
11. ^ Velankar, H. D. (1962.). *_Vr^ttajAtisamuccaya_ of kavi Virahanka*. Rajasthan Oriental Research Institute, Jodhpur. p.101. quote: " For four, variations of meters of two [and] three being mixed, five happens. For five, variations of two earlier – three [and] four, being mixed, eight is obtained.

    In this way, for six, [variations] of four [and] of five being mixed, thirteen happens. And like that, variations of two earlier meters being mixed, seven morae [is] twenty-one. In this way, the process should be followed in all mAtrA-vr^ttas.
12. ^ Knott, Ron. "Fibonacci's Rabbits" (http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html#Rabbits) . University of Surrey Faculty of Engineering and Physical Sciences. http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html#Rabbits.
13. ^ Martin Gardner (1996). *Mathematical Circus*. The Mathematical Association of America. ISBN 0883855062.quote: "It is ironic that Leonardo, who made valuable contributions to mathematics, is remembered today mainly because a 19th-century French number theorist, Edouard Lucas ... attached the name Fibonacci to a number sequence that appears in a trivial problem in Liber abaci." p.153
14. ^ The website [1] (http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibtable.html) has the first 300 F_n factored into primes and links to more extensive tables.
15. ^ Knuth, Donald. "Negafibonacci Numbers and the Hyperbolic Plane" Paper presented at the annual meeting of the Mathematical Association of America, The Fairmont Hotel, San Jose, CA. 2008-12-11 <http://research.allacademic.com/meta/p206842_index.html>
16. ^ Lucas p. 7
17. ^ Weisstein, Eric W., "Binet's Fibonacci Number Formula (http://mathworld.wolfram.com/BinetsFibonacciNumberFormula.html) " from MathWorld.
18. ^ Ball p. 156
19. ^ Following Ball p. 155-156
20. ^ Kepler, Johannes (1966). *A New Year Gift: On Hexagonal Snow*. Oxford University Press. p. 92. ISBN 0198581203. Strena seu de Nive Sexangula (1611).
21. ^ Posamentier, Alfred; Lehmann, Ingmar (2007). *The (Fabulous) FIBONACCI Numbers*. Prometheus Books. p. 305. ISBN 978-1-59102-475-0.
22. ^ M. Möbius, *Wie erkennt man eine Fibonacci Zahl?*, Math. Semesterber. (1998) 45; 243–246.
23. ^ Lucas p. 4
24. ^ Vorobiev, Nikolaĭ Nikolaevich; Mircea Martin (2002). "Chapter 1". *Fibonacci Numbers*. Birkhäuser. pp. 5–6.

ISBN 3-7643-6135-2.

25. ^ [a] [b] [c] Weisstein, Eric W., "Fibonacci Number (http://mathworld.wolfram.com/FibonacciNumber.html) " from MathWorld.
26. ^ Glaister, P.. *Fibonacci power series*. The Mathematical Gazette, 1995, p. 521.
27. ^ Köhler, Günter (February 1985). "Generating functions of Fibonacci-like sequences and decimal expansions of some fractions" (http://www.fq.math.ca/Scanned/23-1/kohler.pdf) (PDF). *The Fibonacci Quarterly* **23** (1): 29–35. http://www.fq.math.ca/Scanned/23-1/kohler.pdf. Retrieved December 31, 2011.
28. ^ Weisstein, Eric W., "Millin Series (http://mathworld.wolfram.com/MillinSeries.html) " from MathWorld.
29. ^ Weisstein, Eric W., "Fibonacci Prime (http://mathworld.wolfram.com/FibonacciPrime.html) " from MathWorld.
30. ^ Ron Knott, "The Fibonacci numbers" (http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibtable.html) .
31. ^ J H E Cohn (1964). "Square Fibonacci Numbers Etc" (http://math.la.asu.edu/~checkman/SquareFibonacci.html) . *Fibonacci Quarterly* **2**: 109–113. http://math.la.asu.edu/~checkman/SquareFibonacci.html.
32. ^ A. Pethő, Diophantine properties of linear recursive sequences II, *Acta Math. Paedagogicae Nyíregyháziensis*, **17**(2001), 81–96.
33. ^ Y. Bugeaud, M. Mignotte, S. Siksek: Classical and modular approaches to exponential Diophantine equations. I. Fibonacci and Lucas perfect powers. *Ann. Math.* (2), **163**(2006), 969–1018.
34. ^ Ross Honsberger *Mathematical Gems III* (AMS Dolciani Mathematical Expositions No. 9), 1985, ISBN 0-88385-318-3, p. 133.
35. ^ Paulo Ribenboim, *My Numbers, My Friends*, Springer-Verlag 2000.
36. ^ Su, Francis E., et al. "Fibonacci GCD's, please." (http://www.math.hmc.edu/funfacts/ffiles/20004.5.shtml) , *Mudd Math Fun Facts*.
37. ^ Paulo Ribenboim (1996), *The New Book of Prime Number Records*, New York: Springer, ISBN 0-387-94457-5, p. 64.
38. ^ Franz Lemmermeyer (2000), *Reciprocity Laws*, New York: Springer, ISBN 3-540-66957-4, ex 2.25–2.28, pp. 73–74.
39. ^ Lemmermeyer, ex. 2.28, pp. 73–74.
40. ^ Lemmermeyer, ex. 2.27 p. 73.
41. ^ Fibonacci and Lucas factorizations (http://mersennus.net/fibonacci/) collects all known factors of F(i) with i<10000.
42. ^ Factors of Fibonacci and Lucas numbers (http://fibonacci.redgolpe.com/) collects all known factors of F(i) with 10000<i<50000.
43. ^ Koshy, Thomas (2007). *Elementary number theory with applications*. Academic Press. p. 581. ISBN 0123724872.
44. ^ Knuth, Donald E. (1997). *The Art of Computer Programming, Volume 1: Fundamental Algorithms* (3rd ed.). Addison–Wesley. ISBN 0-201-89683-4. (p. 343).
45. ^ M. Avriel and D.J. Wilde (1966). "Optimality of the Symmetric Fibonacci Search Technique". *Fibonacci Quarterly* (3): 265–269.
46. ^ Amiga ROM Kernel Reference Manual, Addison–Wesley 1991.
47. ^ IFF – MultimediaWiki (http://wiki.multimedia.cx/index.php?title=IFF#Fibonacci_Delta_Compression) .
48. ^ Zeckendorf representation (http://www.encyclopediaofmath.org/index.php/Zeckendorf_representation) .
49. ^ "Fibonacci Flim-Flam" (http://www.lhup.edu/~dsimanek/pseudo/fibonacc.htm) . http://www.lhup.edu/~dsimanek/pseudo/fibonacc.htm.
50. ^ "Marks for the da Vinci Code: B–" (http://www.cs4fn.org/maths/bee-davinci.php) . *Computer Science For Fun: CS4FN*. http://www.cs4fn.org/maths/bee-davinci.php.
51. ^ Prusinkiewicz, Przemyslaw; James Hanan (1989). *Lindenmayer Systems, Fractals, and Plants (Lecture Notes in Biomathematics)*. Springer-Verlag. ISBN 0-387-97092-4.
52. ^ Vogel, H (1979). "A better way to construct the sunflower head". *Mathematical Biosciences* **44** (44): 179–189. doi:10.1016/0025-5564(79)90080-4 (http://dx.doi.org/10.1016%2F0025-5564%2879%2990080-4)
53. ^ Prusinkiewicz, Przemyslaw; Lindenmayer, Aristid (1990). (http://algorithmicbotany.org/papers/#webdocs) The Algorithmic Beauty of Plants. Springer-Verlag. pp. 101–107. ISBN 978-0387972978. http://algorithmicbotany.org/papers/#webdocs.
54. ^ The Fibonacci Numbers and the Ancestry of Bees (http://www1.math.american.edu/newstudents/shared/puzzles/fibbee.html) .
55. ^ Weisstein, Eric W., "Fibonacci *n*-Step Number (http://mathworld.wolfram.com/Fibonaccin-StepNumber.html) " from MathWorld.

References

- Ball, Keith M. (2003). "Chapter 8: Fibonacci's Rabbits Revisited". *Strange Curves, Counting Rabbits, and Other Mathematical Explorations*. Princeton University Press. ISBN 0691113211.
- Lucas, Édouard (1891). *Théorie des nombres*. **1**. Gauthier-Villars.

External links

- Fibonacci Sequence (http://www.bbc.co.uk/programmes/b008ct2j) on *In Our Time* at the BBC. (listen now (http://www.bbc.co.uk/iplayer/console/b008ct2j/In_Our_Time_Fibonacci_Sequence))
- Sloane's A000045 : Fibonacci Numbers (http://oeis.org/A000045) . The On-Line Encyclopedia of Integer Sequences. OEIS Foundation.
- Periods of Fibonacci Sequences Mod m (http://www.mathpages.com/home/kmath078/kmath078.htm) at MathPages
- Scientists find clues to the formation of Fibonacci spirals in nature (http://www.physorg.com/news97227410.html)
- Implementation to calculate Fibonacci sequence in Lisp (http://wikinternet.com/wordpress/code/lisp/fibonacci-number/)

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