Calculation of the Focal Length of an Offset Satellite Dish Antenna, Revisited

by A33 (my alias on several satellite-forums), the Netherlands
version 1.0, July 2020.

Part A: Calculating Focal length, for offset dishes with a *flat* dish face

Maybe you know the article *Calculation of the Focal Length of an Offset Satellite Dish Antenna* by John Legon, written about 8 years ago I think, in which he describes the 'simple' equation he derived for calculating the focal length of an offset paraboloid dish, using as the (only) required input variables: height and width of the dish, and maximum depth.
Also, he gives very helpful equations for the top string and bottom string lengths (the distance from focal point to top of the dish, and that to bottom of the dish). Using these measures, the exact feedhorn location can be determined and/or checked.

In a spreadsheet calculator that I made for calculating various dish specifications of offset paraboloid dishes, allowing various sets of input variables for the calculations, I do however not prefer the method that uses the maximum depth of the dish as input variable, but the method that uses **depth at the center of the dish**. The measuring of the depth at the center I find easier, as you don't have to find the exact location of the deepest depth first.
However, *calculating* the focal length out of the depth at the center (combined with width and height of the dish, as the other needed input variables) used to be a bit more difficult. I till now used a step by step calculation, first defining the coordinates of the three points of the dish, relative to the offset angle; and then solving a quadratic equation on those points. I didn't know a faster or easier way to calculate on those inputs.

On the basis of a specific property of paraboloid dishes that I discovered not long ago, I could derive a new equation for calculating the focal length of an offset paraboloid dish, using the depth at the center as input:

A33 focal length equation for offset paraboloid dishes, using depth at the center:

$$f = \frac{W^3 d}{4H} * \left(\frac{1}{4d^2} + \frac{1}{H^2} - \frac{1}{W^2} \right) \qquad \text{(equation 1)}$$

(f=focal length, W=width of dish, d=depth of dish at the center, H=height of dish.)

Though the equation might not be as simple as is John Legon's, it has the advantage that you can 'simply' measure depth at the center of the dish, next to height and width of the dish, and don't have to solve any quadratic equation anymore.

Special property of paraboloid dishes

The derivation of the equation is based on a specific property of a paraboloid dish, that I discovered only recently, and that I had never read about (though I don't think I would be the very first, to discover it?). I'll first describe what I found out, and then the derivation on the base of that.

To start: we all know the basic formula to calculate the focal length of a prime focus paraboloid *(where D=diameter of the paraboloid, d=depth of the paraboloid at the center (which is also where the deepest point is))*:

$$f = \frac{D^2}{16d} \qquad \text{(equation 2)}$$

I found, that this equation to get the proper focal length does not only apply to the depth and width measurement *at the center* of the prime focus paraboloid, but that it applies to the measurement **at any height of the paraboloid**; as long as the measurements of width and depth are taken **in a plane parallel to the symmetry axis** of the paraboloid, and as long as the width is (as usual) measured perpendicular to the symmetry axis.
(The symmetry axis, of course, is the line through focal point and vertex of the paraboloid.)

In other words, as an example: If the symmetry axis of a paraboloid is a horizontal line, you can take **any** horizontal slice out of the paraboloid, and with width and depth measures of that slice you can calculate the one and only focal length of that paraboloid.

This specific property applies not only to prime-focus dishes, but also to offset dishes, because such a parallel slice can be taken, regardless of the dish face (offset) angle.

In figure 1 the principle of this paraboloid property is drawn; showing three horizontal slices through an offset dish as examples.
(Likewise, you could draw a vertical dish face, for instance at the front of slice 1, and find the same results for various slices of a prime focus dish with that dish face.)

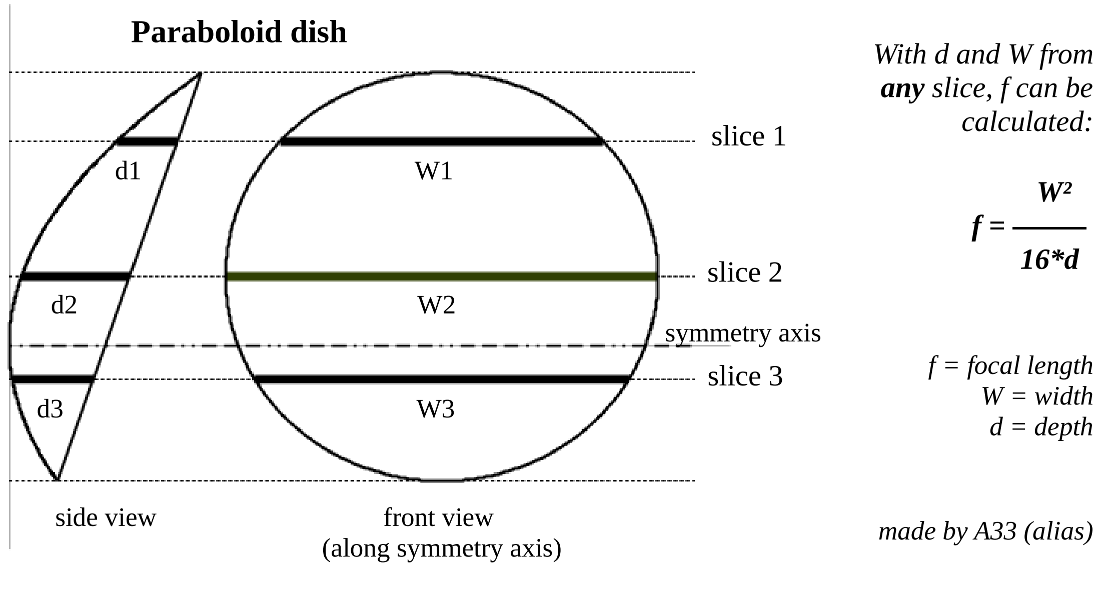

*Figure 1: Special property of slices of a paraboloid dish, that are taken parallel to the symmetry axis: focal length of the paraboloid can be calculated as f = width² / (16 * depth) of* ***any*** *slice.*

Derivation of this new focal length equation

When we want to apply this special property of paraboloid dishes to the input variables of width (*W*) and depth (*d*) at the center of an offset dish, a problem arises. The depth measurements of offset dishes are normally taken *perpendicular to the dish face*, so not parallel to the symmetry axis. So, these measurements must be 'transformed' to **new width and depth values, that do lie in a plane, parallel to the symmetry axis**. Thus, instead of equation 2, we get (*W'=new width, d'=new depth*):

$$f = \frac{(W')^2}{16d'} \qquad \text{(equation 3)}$$

For the derivation, we can use the plane through the center point (C) on the dish surface (so with equally long distance, measured in straight lines, to top and to bottom of the dish).
For an indication of the relevant lines in the derivation, see figure 2.

First, we will calculate the new depth (d', parallel to the symmetry axis) from the measured depth (d) at the dish center. That can be found, using the offset angle (where *d=measured depth at the center, offset=offset angle*):

$$\cos(offset) = \frac{d}{d'} \text{ , so } d' = \frac{d}{\cos(offset)} \qquad \text{(equation 4)}$$

Now, as *cos (offset)* on a flat-faced paraboloid dish can always be calculated by *width / height*, we can find the new width value with the following equation (where *W=width, H=height of the offset dish*):

$$d' = \frac{d}{W/H} \text{ , equals } \boldsymbol{d' = \frac{dH}{W}} \qquad \text{(equation 5)}$$

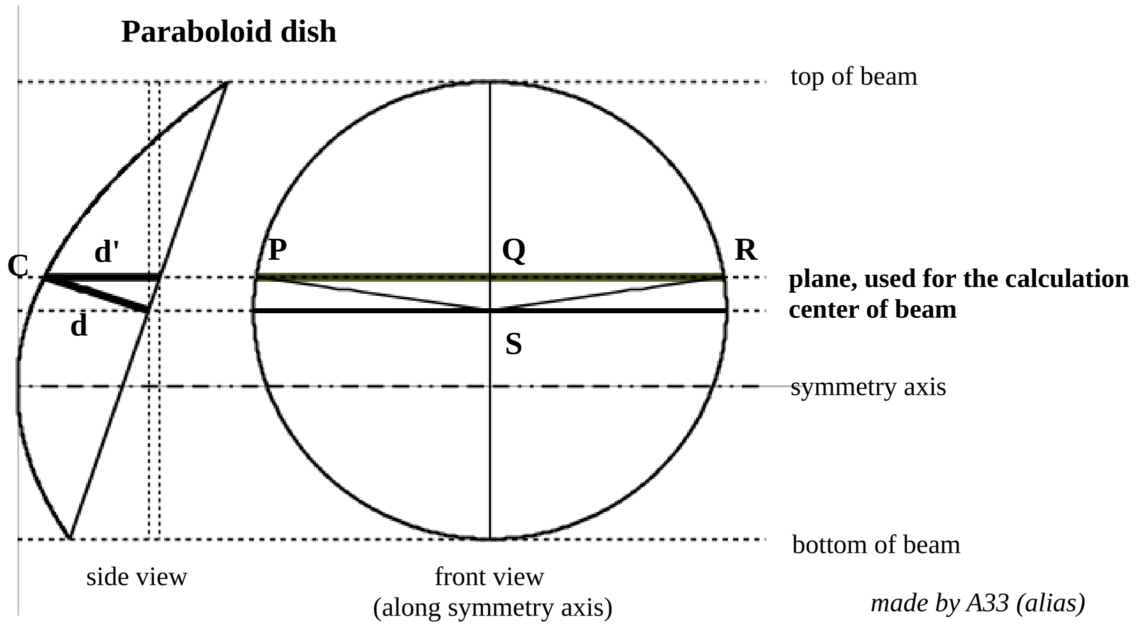

Figure 2: Showing the relevant lines that are used deriving 'new depth' and 'new width' values (for focal length calculation of an offset paraboloid dish).

Second, we will derive the 'new width' at the relevant plane; which is (in other words) the new 'chord length' at the dish face.

As we know, the dish face forms an ellipse. But because an ellipse can be seen as a projected circle, we can more easily calculate this new chord length as if it lies on the projected circle of the dish face, which in this case is exactly perpendicular to the symmetry axis of the paraboloid.

The new chord length (see again figure 2) is distance PR, so twice the distance QR.
From Pythagoras we know:

$$RS^2 = QR^2 + QS^2, \text{ so } QR^2 = RS^2 - QS^2 \qquad \text{(equation 6)}$$

And we know:

$$RS = circle\ \ radius = W/2 \qquad \text{(equation 7)}$$

$$QS = d * \sin(offset) \qquad \text{(equation 8)}$$

$$QR = (W')/2 = half\ of\ the\ new\ width \qquad \text{(equation 9)}$$

Substituting these variables in equation 6, and multiplying everything by 4, we get:

$$(W')^2 = W^2 - 4 * d^2 * \sin^2(offset) \qquad \text{(equation 10)}$$

Now, we know as a general rule that $sin^2(x) = 1 - cos^2(x)$,
and that *cos(offset) = width / height*,
so the previous equation can be transformed to the equation, that defines the new width:

$$\boldsymbol{(W')^2 = W^2 - 4d^2\left(1 - W^2/H^2\right)} \qquad \text{(equation 11)}$$

Finally, entering the outcomes of equations 5 and 11 in equation 3, we get:

$$f = \frac{W^2 - 4d^2\left(1 - W^2/H^2\right)}{16dH/W} \qquad \text{(equation 12)}$$

Simplifying it one step further, we get a formula that looks a bit like John Legon's equation, with an extra correction factor for the depth measurement at another location:

$$f = \frac{W^3 - 4Wd^2\left(1 - W^2/H^2\right)}{16dH} \qquad \text{(equation 13)}$$

I prefer, however, to derive this equation one step further, to the formula that is given in equation 1 (though other forms for the equation are also possible):

$$\boldsymbol{f = \frac{W^3 d}{4H} * \left(\frac{1}{4d^2} + \frac{1}{H^2} - \frac{1}{W^2}\right)}$$

(A33 focal length equation for offset paraboloid dishes, using depth at the center)

This equation gives exactly the same outcome as the solving procedure of the three points quadratic equation for the paraboloid, as would be expected.

Some additional considerations

1. All the above mentioned dish measurements assume a 'normal' higher-than-wide offset dish, with a flat dish face, with the measurements done at the flat dish face. For paraboloid dishes with a non-flat dish face, that are for instance wider-than-high, another approach is needed (see part B, below).

2. Be aware that the focal length (the outcome of the equation) is defined as the distance from vertex to focal point of the paraboloid, so it is the focal length of the 'parent prime-focus dish'. When you need to know the 'effective focal length' from the offset dish (the length of the bisector

line from focal point to 'G-spot', as it is sometimes called on the internet), for instance for multifeed distance calculations, that can be calculated from the focal length in combination with other dish specifications.

3. The special property of the paraboloid dish, mentioned above, can also easily be used for the derivation of the John Legon equation that is using maximum depth as input variable.

Starting with equation 3, and using the plane that lies at the deepest point of the dish as the relevant plane for the calculation:
a. the 'new width' is exactly the (maximum) width of the dish *(W)*, and
b. the 'new depth' in the relevant plane would be: *(maximum depth) / (cos (offset))*.
Applying the steps in equations 4, 5, and 13, the end result becomes the already known equation:

$$\textit{Focal length} = \textit{width}^3 / (16 * \textit{maximum depth} * \textit{height}) \qquad \text{(equation 14)}$$

(Legon's equation for the focal length of an offset dish antenna)

4. Notice that for prime focus dishes, where height of dish is equal to width of dish *(W=H)*, and maximum depth is equal to depth at the center, both above equations (1 and 14) to calculate the focal length can also be applied, though they are unnecessarily complicated.
They can both be simplified, to the widely known form: $f = W^2 / (16d)$.

Part B: Calculating Offset angle and Focal length, for offset dishes with *non-flat* dish face

If you want to use equation 1 in this article for calculating the focal length of an offset paraboloid dish that has a ***non-flat dish face***, and you already have the measures for (1) dish height and (2) dish depth (relative to the dish height measure); you would need to find the dish width, corresponding to that height and depth measure, to be able to calculate the proper dish's offset angle.
Legon describes a way to measure that width, using the 'water method'.

That width, and the dish's offset angle, can however also be calculated with two extra measurements directly on the dish, without using the water method. The extra needed measurements are (3) dish width (measured exactly above the dish center), and (4) dish depth at the center, *relative to the width measurement*.

So, the needed input variables are (see figure 3):
a. distance DE, height of the dish, from rim to rim (called *'Hh'* further on);
b. distance CA, depth of the dish at the center, relative to the height measure (*'dh'* further on);
c. distance FG, width of the dish, from rim to rim, measured above the center (*'Ww'* further on);
d. distance CB, depth of the dish at the center, relative to the width measure (*'dw'* further on).
The wanted measure would be *'Wh'*: the corresponding width to the 'height'-dish front; which is measured perpendicular to the side view drawing, through point A.

(Notice that *W*, *H* and *d* in this part of the article always have the indication *w* or *h* added to them! *Hh*, *Wh* and *dh* measures relate to the imaginary flat-faced *'height'* dish front, *Ww* and *dw* measures relate to the imaginary flat-faced *'width'* dish front.)

N.B.: The width measure (FG) is not necessarily at the widest point of the dish, as the dish design of a non-flat dish can choose its widest width at any height of the dish. To find the proper location for

F and G above the center, remember that DF=FE=EG=GD, in the (three-dimensional) diamond shape of DFEG.
N.B. 2: In figure 3, the circle through points F and G is the *projected circle* of the *imaginary* dish face, for a dish that has *F and G* at its dish face. The smaller circle is the projected circle of the imaginary dish face for a dish that has D and E at its dish face.

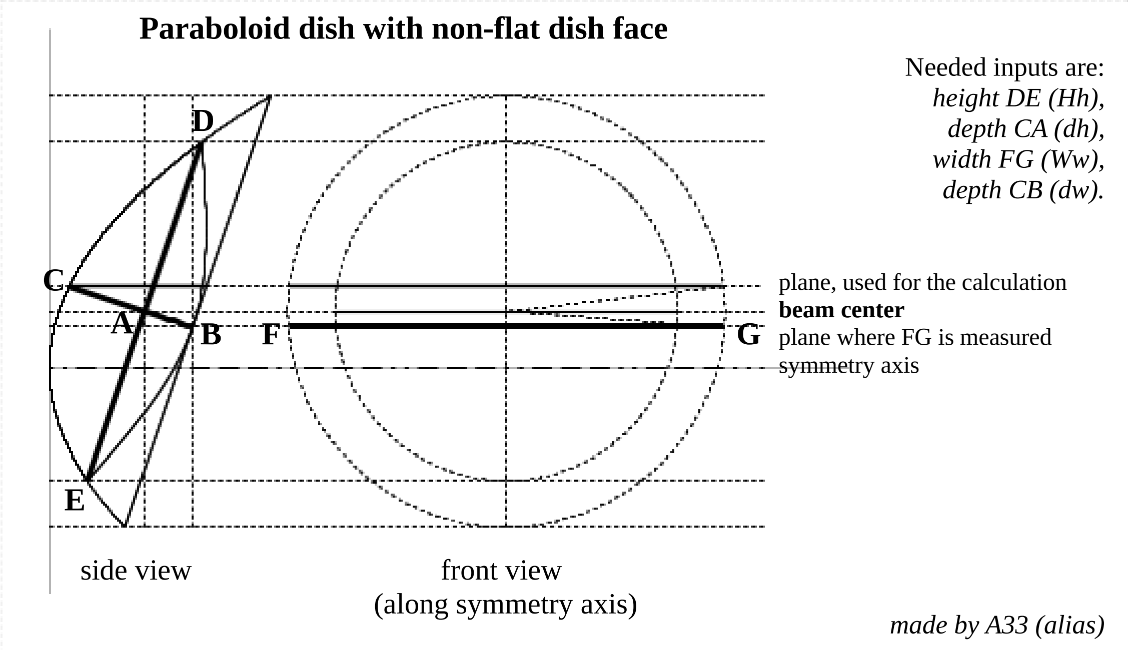

Figure 3: Indicating the measurements that are needed for non-flat paraboloid offset dishes, and some lines relevant for the derivation of the offset-angle equation for those dishes.

Using the above input measurements, I derived two basic equations for the non-flat dish face dish calculations on the basis of those input measures: one for **calculating the *offset angle***, and another one for **calculating the *corresponding width (Wh)*** as mentioned above, giving the needed input for the focal length calculation of equation 1 in part A of this article.

The equation to get the offset angle ('*offset*') is *(for the meaning of the variables, see above*):

$$\boldsymbol{offset} = \mathbf{arccos} \sqrt{\frac{\boldsymbol{Ww^2\, dh + 4\, dw\, dh\, (dw - dh)}}{\boldsymbol{Hh^2\, dw + 4\, dw\, dh\, (dw - dh)}}} \qquad \text{(equation 15)}$$

(A33 equation for the offset angle of a non-flat paraboloid offset dish)

From this offset angle equation, the corresponding width value (*Wh*) to the *Hh* and *dh* measurements can easily be calculated, using *cos(offset) = Wh/Hh* :

$$Wh = Hh * \sqrt{\frac{Ww^2\, dh + 4\, dw\, dh\, (dw - dh)}{Hh^2\, dw + 4\, dw\, dh\, (dw - dh)}} \qquad \text{(equation 16)}$$

Starting with these above equations on the non-flat dish face dishes, allows you to calculate most other dish specifications of those dishes, using the equations for 'normal' (flat dish face) dishes again.

Derivation of the non-flat dish face offset angle equation

For the derivation of the equation, I used the special property of paraboloid dishes, that is described in part A of this article. As the relevant plane for the calculation, I again chose the plane through the center of the dish surface (*C*), parallel to the symmetry axis.

For the input measurements ***Hh* and *dh***, we can use equation 4 and 10 to 'fill' equation 3, so we get:

$$f = \frac{Wh^2 - 4\,dh^2 \sin^2(offset)}{(16\,dh)\ /\ \cos(offset)} \qquad \text{(equation 17)}$$

For the input measurements ***Ww* and *dw***, equation 4 also applies, but to get an equation for *W'* out of the *Ww* measurement, a different approach is needed.

For that we use Pythagoras twice:

$$(W'/2)^2 = \{\ (Ww/2)^2 + [(dw-dh)\sin(offset)]^2\ \} - [dh\ \sin(offset)]^2 \qquad \text{(equation 18)}$$

Which can be rearranged to:

$$(W')^2/4 = Ww^2/4 + \sin^2(offset)\ (dw^2 - 2dwdh + dh^2 - dh^2) \qquad \text{(equation 19)}$$

Simplifying this equation a bit, we get for $(W')^2$:

$$(W')^2 = Ww^2 + 4\sin^2(offset)\ \ dw\ (dw - 2dh) \qquad \text{(equation 20)}$$

So the focal length equation for the input measurements *Ww* and *dw* is (using eq. 3, 4 and 20):

$$f = \frac{Ww^2 + 4\sin^2(offset)\ dw\ (dw - 2dh)}{(16\,dw) / \cos(offset)} \qquad \text{(equation 21)}$$

Now we can combine the focal length equations 17 and 21, **for both** the *Hh* and *dh* measurements, and the *Ww* and *dw* measurements:

$$f = \frac{Wh^2 - 4dh^2 \sin^2(offset)}{(16dh)\ /\cos(offset)} = \frac{Ww^2 + 4\sin^2(offset)\ dw\ (dw - 2dh)}{(16dw)\ /\cos(offset)} \qquad \text{(equation 22)}$$

We'll have to do a bit of rearranging:

$$dw\ [\ Wh^2 - 4dh^2\sin^2(offset)\ ] = dh\ [\ Ww^2 + 4\sin^2(offset)\ dw\ (dw - 2dh)\ ] \qquad \text{(equation 23)}$$

$$dwWh^2 - 4dwdh^2\sin^2(offset) = dhWw^2 + 4\sin^2(offset)\ dwdh(dw - 2dh) \qquad \text{(equation 24)}$$

$$dw\ Wh^2\ -\ 4\sin^2(offset)\ dw\ dh\ (dh + dw - 2dh) = dhWw^2 \qquad \text{(equation 25)}$$

As $\sin^2(x) = 1 - \cos^2(x)$, or $-\sin^2(x) = \cos^2(x) - 1$:

$$dw\ Wh^2 + 4\cos^2(offset)\ dw\ dh\ (dw–dh) - 4\ dw\ dh\ (dw–dh) = dh\ Ww^2 \qquad \text{(equation 26)}$$

And because $Wh^2 = Hh^2 \cos^2(offset)$:

$$dw\ Hh^2\cos^2(offset) + 4\cos^2(offset)\ dw\ dh\ (dw–dh) - 4\ dw\ dh\ (dw–dh) = dh\ Ww^2 \qquad \text{(equation 27)}$$

Rearranging further:

$$\cos^2(offset)\,[dw\,Hh^2 + 4\,dw\,dh\,(dw-dh)] = dh\,Ww^2 + 4\,dw\,dh\,(dw-dh) \qquad \text{(equation 28)}$$

$$\cos^2(offset) = \frac{Ww^2\,dh + 4\,dw\,dh\,(dw-dh)}{Hh^2\,dw + 4\,dw\,dh\,(dw-dh)} \qquad \text{(equation 29)}$$

And so finally we get equation 15:

$$\boldsymbol{offset = \arccos\sqrt{\frac{Ww^2\,dh + 4\,dw\,dh\,(dw-dh)}{Hh^2\,dw + 4\,dw\,dh\,(dw-dh)}}}$$

(A33 equation for the offset angle of a non-flat paraboloid offset dish)

The result of this equation is exactly the same as that of the iterative algorithm (from a starting estimate, with many repeated corrective steps, reaching the actual value in the end), that I used previously in my calculator. This equation offers a much quicker and easier way to calculate.

Some additional considerations

1. Starting with equation 22, you could also come to other forms for the equation, that of course result in the same outcome. Personally, I liked this form best, because of the repeated term in the numerator and denominator, which makes it a little bit easier to remember I think.

2. Note: the equations in this part (part B) of this article apply to paraboloid dishes with a non-flat dish face, that are parabolic in all dimensions (both horizontally and vertically).
They don't apply to other dishes with a non-flat dish face, that are built for multifeed reception (e.g. toroid dishes), which are usually parabolic vertically, but circular horizontally. For those, another approach is needed; I haven't finished my work calculating on those.
When I find 'simple' equations (instead of more elaborate algorithms) for those multifeed/toroid dishes, I'll probably add a part C to this document.

After that, I will simplify my spreadsheet calculator for offset dishes (that allows using various sets of input variables) by incorporating the newly found equations of this article; and hopefully finish that project also.

(A33 is my alias on several dutch-, english-, german- and italian-spoken satellite forums.)

= =